Credit Risk Modeling with Delayed Information

Takanori Adachi
Ryozo Miura
Hidetoshi Nakagawa

Graduate School of International Corporate Strategy
Hitotsubashi University

March 10, 2012
CREST and 4th Ritsumeikan-Florence Workshop
on Risk, Simulation and Related Topics
APU, Beppu, Japan

Structural Credit Models and their Problem

Structural versus Reduced-Form

One of the important types of credit risk models is a *structural* model attempting to explain the mechanism by which default takes place.

On the other hand, another important type of credit risk models known as a *reduced-form* leaves the precise mechanism leading to default unspecified.

Our interest is on the mechanism and its influence to an increasing information system by which several pricing will be made.
Therefore, we focus on structural models in this research.

In the following, we see the framework of structural models initiated by Merton, and examine its problem.

A problem of structural credit risk models

Structural approach in credit risk modeling theory based on the default time is defined by

$$\tau := inf\{t > 0 \mid V_t \leq L\}$$

where V_t is the firm value at time $t \geq 0$, and L is a liability with $L < V_0$.

If we have complete information on V_t in a real time base, the theoretical credit spread of a defaultable zero-coupon bond converges to 0 as the time goes to its maturity date.

Why? Because the default time becomes a predictable stopping time.

However, empirically observing credit spread does not become 0 even at just before its maturity time.

Incomplete Models

We need a kind of incompleteness on the information about V_t in the model, that makes its default time a *totally inaccessible* stopping time.

Many authors have tried several formulation to make this happen.

1. Duffie and Lando (2001) introduce noise into the market's information set. Observe the firm's asset value plus noise at equally spaced, discrete (non-continuous) time points.
2. Kusuoka (1999) extends Duffie and Lando's model to continuous time observations.
3. Nakagawa (2001) presents a filtering model of a default time in a rigid mathematical setting.
4. Çetin, Jarrow, Protter and Yildirim (2004) reduce the information the market can see instead of appending noise.
5. Giesecke's model (2006) makes the default barrier be unobservable to the market.

Market Delay Models

One of the ways to make the model incomplete is to introduce a *delay* of the information reaching to some parities as a source of the model incompleteness.

1. Lindset, Lund and Persson (2008) introduce a model having constant lags for both managers and markets.
2. Guo, Jarrow and Zeng (2009) propound a model having stochastic delay based on an increasing sequence of stopping times.

Our Goal

We adopt the market delay approach and would like to investigate

1. What is a *natural* definition of randomly delayed time?
2. What kind of filtrations can we get through the randomly delayed time?
3. How do we calculate conditional expectations given the filtration in order to make valuation?

Information Delay

Information Delay

What is the market delay? (1)

First, we give a formulation of *market delay*.

Suppose that we are market participants, and observe an event at time t. The event, however, happened at time m_t that may be ahead of t. That is,

$$m_t \leq t.$$

In other words, there is a delay

$$t - m_t$$

for market participants to know the event.

What is the market delay? (2)

It is also natural to assume the following monotonicity to preserve the order of causality:

$$s < t \text{ implies } m_s \le m_t.$$

On the other hand, insiders know the occurrence of the event without delay.

Assumption

Throughout this presentation, all the discussion is under the filtered probability space

$$(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}),$$

where the filtration $\mathbb{F}$ satisfies the usual condition.

GJZ model

Guo, Jarrow, and Zeng treats m_t as the following stochastic process.

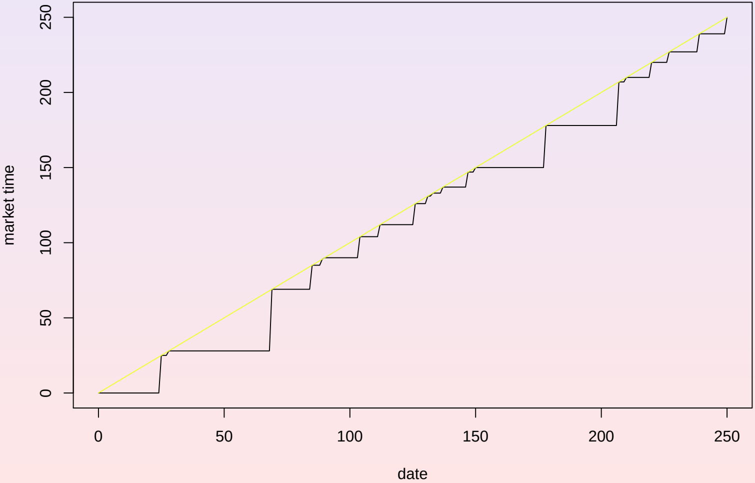
Definition

[Time Change Process (Guo, Jarrow, Zeng, 2009)]
An $\mathbb{F}$-*time change process* is a stochastic process $m = \{m_t\}_{t \geq 0}$ satisfying

1. $m_0 = 0, \ a.s.,$
2. $m_t \leq t, \ a.s.$ for all $t \geq 0$,
3. $m_s \leq m_t, \ a.s.$ for all $s < t$,
4. Each m_t is an $\mathbb{F}$-stopping time.

Poisson Market Time

Renewal market time having an exponential interval distribution $Exp(10)$.



Renewal Market Time

One of the natural examples of randomly delayed time is a *renewal* market time.

Definition

[Renewal Market Time]

1. $X_n \sim$ i.i.d. random variables such that $0 < \mathbb{E}^{\mathbb{P}}[X_n] < \infty$ for $n = 1, 2, \ldots,$
2. $S_n := \sum_{k=1}^{n} X_k,$
3. $N_t := \sup\{n \mid S_n \leq t\},$
4. $m_t := S_{N_t}.$

Can Renewal Market time be a model of GJZ time change process?

No!

since the fourth condition " m_t is a stopping time " is too strong.

Market Times

Raw Market Time

We introduce a wider family of stochastic processes than GJZ's time change processes.

Definition

[Raw Market Time]

A *raw market time* is a stochastic process $m = \{m_t\}_{t \geq 0}$ satisfying

1. $m_0 = 0, \ a.s.,$
2. $m_t \leq t, \ a.s. \quad \text{for all } t \geq 0,$
3. $m_s \leq m_t, \ a.s. \quad \text{for all } s < t.$

$\mathbb{F}$-Market Time

Definition

[$\mathbb{F}$-Market Time] An $\mathbb{F}$-*market time* is a raw market time which is $\mathbb{F}$-adapted.

Idempotent Market Time

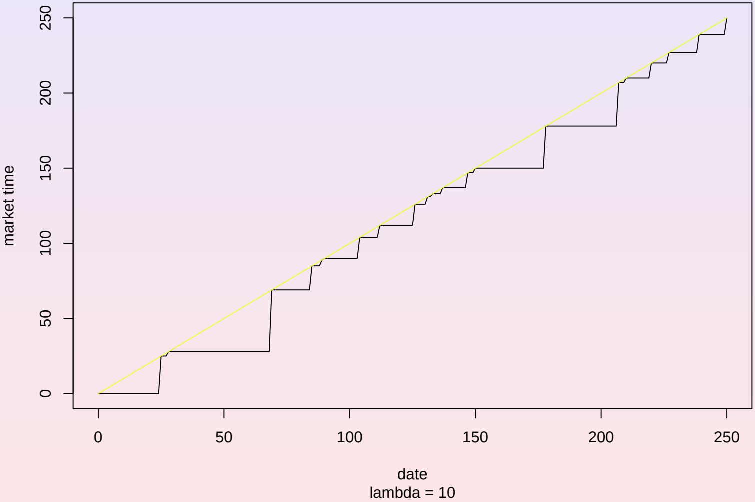
Definition

[Idempotent Market Time]
A raw market time $m = \{m_t\}_{t \geq 0}$ is called *idempotent* if

$$m_{m_t} = m_t, \ a.s. \quad \text{for every} \quad t \geq 0.$$

Identity market time and renewal market times are idempotent.

Poisson Market Time (reproduced)



GJZ model does not accept Idempotent Market Time

Proposition

Let $m = \{m_t\}_{t \geq 0}$ be an idempotent raw market time where m_t is an $\mathbb{F}$-stopping time. Then, for every pair t and s with $t \geq s$, we have

$$\{m_t = m_s\} \in \mathcal{F}_s.$$

This proposition says that at the current time s we are able to know if the information will have increased since now by any future time t, which is not realistic.

This is why we exclude the condition of m_t being a stopping time from the definition of market times.

Characterization of Idempotent Raw Market Times

Definition

For a random set $M \subset \mathbb{R}_+ \times \Omega$, define a process $m^M : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ by

$$m_t^M(\omega) = \sup\{s \leq t \mid (s, \omega) \in M\},$$

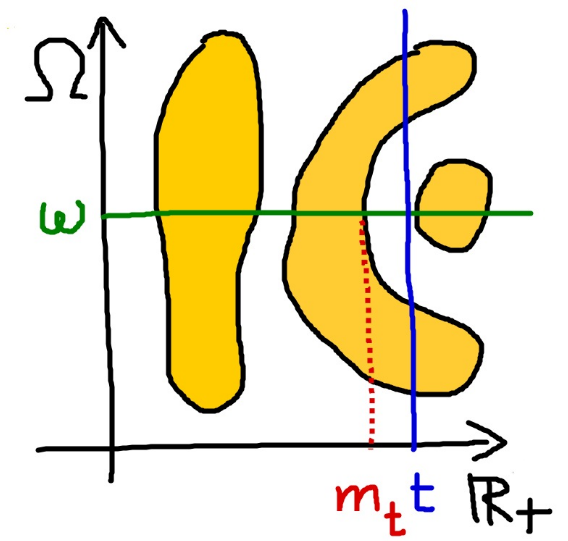
where we use the convention $\sup \emptyset = 0$.

Theorem

Let $m : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be a process. Then, m is an idempotent raw market time iff there exists a random set $M \subset \mathbb{R}_+ \times \Omega$ such that

$$m = m^M.$$

$M \subset \mathbb{R}_+ \times \Omega$

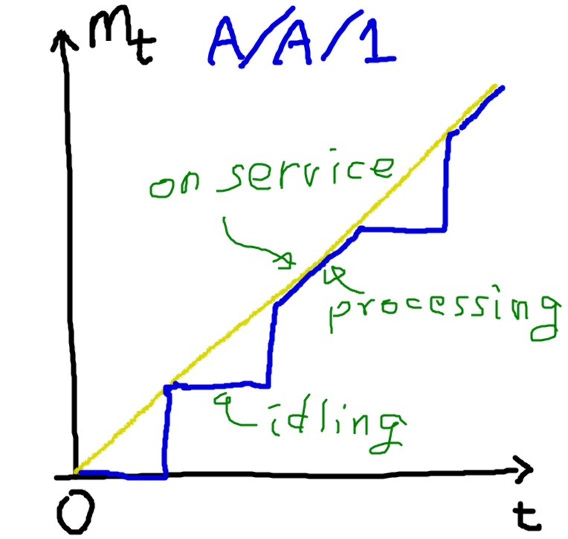


Characterization of Idempotent $\mathbb{F}$-Market Times

Theorem

Let $m : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ *be a process. Then,* m *is an idempotent* $\mathbb{F}$*-market time iff there exists an* $\mathbb{F}$*-optional set* $M \subset \mathbb{R}_+ \times \Omega$ *such that* $m = m^M$*.*

Idempotent Market Times and Queuing Thoery - One-Wicket System



Explicit Representation of Idempotent Market Times

Theorem

Let m be an idempotent $\mathbb{F}$-market time whose sample paths are càdlàg. Define a generalized function X_t by

$$X_t := \mathbb{1}_{I^m - J^m}(t) + \sum_{u \in J^m} (m_u - m_{u-})\delta(t-u)$$

where I^m and J^m are random sets defined by

$$I^m := \{t > 0 \mid m_t = t\} \quad \textit{and} \quad J^m := \partial I^m,$$

and δ is the Dirac delta function. Then, we have for every $t \in \mathbb{R}_+$,

$$m_t = \int_0^t X_s ds.$$

Characterization of Honest Times

Theorem

A random time $\tau : \Omega \to \bar{\mathbb{R}}_+$ *is* $\mathbb{F}$*-honest if and only if there exists an idempotent* $\mathbb{F}$*-market time* m *such that for every* $t > 0$, $\tau = m_t$ *on* $\{\tau \leq t\}$, *i.e.* $\tau = m_\infty$.

Market Filtrations

Continuously Delayed Filtration

We now turn our focus on the filtration generated by delayed times.

First, let us see the continuously delayed filtrations introduced by Guo, Jarrow and Zeng.

Definition

[Continuously Delayed Filtration(Guo, Jarrow, Zeng, 2009)]
Let $m = \{m_t\}_{t \geq 0}$ be a $\mathbb{F}$-time change process. Then, the *continuously delayed filtration* $\mathcal{F}^{GJZ}_{m_t}$ is defined by

$$\mathcal{F}^{GJZ}_{m_t} := \{A \mid (\forall s \geq 0) A \cap \{m_t \leq s\} \in \mathcal{F}_s\}.$$

This definition works since m_t is an $\mathbb{F}$-stopping time.

Market Filtration

Definition

[Market Filtrations] Let $m = \{m_t\}_{t \geq 0}$ be an $\mathbb{F}$-market time. The *market filtration* modulated by m is the filtration $\mathbb{F}^m = \{\mathcal{F}_t^m\}_{t \geq 0}$ defined by for $t \geq 0$,

$$\mathcal{F}_t^m := \bigvee_{0 \leq s \leq t} \mathcal{F}_{m_s},$$

where

$$\mathcal{F}_{m_t} := \sigma\{Z_{m_t} \mid Z = \{Z_t\}_{t \geq 0} \text{ is an } \mathbb{F}\text{-optional process. }\}.$$

Theorem

Let $m = \{m_t\}_{t \geq 0}$ be an $\mathbb{F}$-market time. Then, $\mathbb{F}^m$ is a subfiltration of $\mathbb{F}$.

Relation between Market Filtrations and Continuously Delayed Filtration

We are interested in the case when it happens to have the market time m_t being a stopping time for every t.

Theorem

Let $m = \{m_t\}_{t \geq 0}$ be an $\mathbb{F}$-market time where each m_t is an $\mathbb{F}$-stopping time. Then, $\mathcal{F}^m_t = \mathcal{F}^{GJZ}_{m_t}$.

Market Filtrations Generated by Idempotent Market Times

Theorem

Let m be an idempotent $\mathbb{F}$-market time. Then, for every $t \geq 0$, we have $\mathcal{F}_t^m = \mathcal{F}_{m_t}$.

This makes the calculation of the following conditional expectation much easier.

In some sense, this is another evidence that the *idempotent*ness is a useful attribute for classifying market times.

Market Times in a Binomial Model

Market Times in a Binomial Model

What we want to do is ...

$$
\begin{aligned}
D(t,T) &= \mathbb{E}^{\mathbb{P}}\left[e^{-(T-t)r}\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t^m\right] \\
&= e^{-(T-t)r}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[0,T]} V_u>L\}} \mid \mathcal{F}_{m_t}\right] \\
&= e^{-(T-t)r}\mathbb{1}_{\{\inf_{u\in[0,m_t]} V_u>L\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[m_t,T]} V_u>L\}} \mid \mathcal{F}_{m_t}\right] \\
&= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[m_t,T]} Y_u>0\}} \mid \mathcal{F}_{m_t}\right] \\
&= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[m_t,T]} Y_u>0\}} \mid m_t, Y_{m_t}\right] \\
&= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\mathbb{P}\Big(\inf_{u\in[m_t,T]} Y_u>0 \mid m_t, Y_{m_t}\Big) \\
&= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\left[\Phi\left(\frac{Y_{m_t}+\nu(T-m_t)}{\sigma\sqrt{T-m_t}}\right)\right. \\
&\qquad \left. - e^{-2\nu Y_{m_t}\sigma^{-2}}\Phi\left(\frac{-Y_{m_t}+\nu(T-m_t)}{\sigma\sqrt{T-m_t}}\right)\right].
\end{aligned}
$$

Time

Definition

[Time] Let δ be a given positive number.

1. $\mathcal{T} := \{n\delta \mid n = 0, 1, 2, \ldots, N\}$,
2. The *horizon* is the number $T := N\delta$,
3. For $s, t \in \mathcal{T}$, $[s, t]_{\mathcal{T}} := \{u \in \mathcal{T} \mid s \leq u \leq t\}$. Similarly, we define $[s, t[_{\mathcal{T}}$, $]s, t]_{\mathcal{T}}$ and $]s, t[_{\mathcal{T}}$,
4. $\mathcal{T}_+ :=]0, T]_{\mathcal{T}}$,
5. For $t \in \mathcal{T}_+$, $t- := t - \delta$,
6. For $t \in [0, T[_{\mathcal{T}}$, $t+ := t + \delta$.

Measurable Space

Definition

[Measurable Space] In the following, $\mathfrak{H}$, $\mathfrak{T}$ and $\perp$ are distinct constants.

1. $\Omega := \{\mathfrak{H}, \mathfrak{T}\}^{\mathcal{T}_+}$.
   For $\omega \in \Omega$, we expand its domain to $\mathcal{T}$ by defining $\omega(0) := \perp$,
2. For $t \in \mathcal{T}$, the binary relation $\sim_t$ on Ω is defined by for $\omega, \omega' \in \Omega$,
   $$\omega \sim_t \omega' \;\Leftrightarrow\; (\forall s \in]0, t]_{\mathcal{T}})\omega(s) = \omega'(s),$$
3. For $t \in \mathcal{T}$, $\mathcal{F}_t := \sigma(\Omega/ \sim_t)$,
4. $\mathcal{F} := \mathcal{F}_T$.

Probability

Definition

[Probability] Let $p \in]0,1[$ be a given number.

1. $\mathbb{P} : \Omega \to [0,1]$ is defined by for $\omega \in \Omega$, $\mathbb{P}(\omega) := p^{\#\omega}(1-p)^{N-\#\omega}$ where $\#\omega$ is the cardinality of $\omega^{-1}(\mathfrak{H})$,
2. $\mathbb{P} : \mathcal{F} \to [0,1]$ is defined by for $A \in \mathcal{F}$, $\mathbb{P}(A) := \sum_{\omega \in A} \mathbb{P}(\omega)$.

Neighborhood I

Definition

For a random time τ, a *neighborhood* of $\omega \in \Omega$ at τ is the set

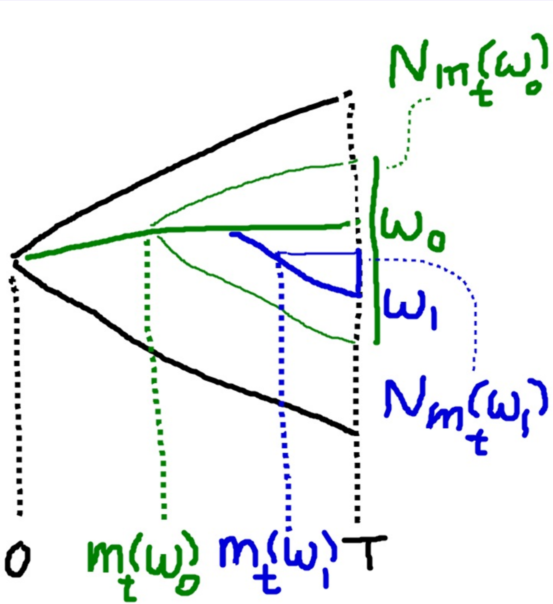
$$N_\tau(\omega) := [\omega]_{\sim_{\tau(\omega)}}.$$

Theorem

Let m be an idempotent $\mathbb{F}$-market time.

$$\mathcal{F}_t^m = \sigma\{N_{m_t}(\omega) \mid \omega \in \Omega\}.$$

Neighborhood II



Conditional Expectation

Theorem

Let m be an idempotent $\mathbb{F}$-market time, Y be a random variable and X be an $\mathcal{F}_{m_t}$-measurable random variable. Then,

$$\mathbb{E}^{\mathbb{P}}[Y \mid \mathcal{F}_t^m] = X \quad \textit{iff} \quad (\forall \omega_0 \in \Omega)(\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} Y] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} X]).$$

Process Y

Definition

[Process Y]

1. For $t \in \mathcal{T}_+$, a Bernoulli process X is defined by

$$X_t(\omega) = \begin{cases} \sqrt{\delta} & \text{if } \omega(t) = \mathfrak{H} \\ -\sqrt{\delta} & \text{if } \omega(t) = \mathfrak{T}. \end{cases}$$

2. For $t \in \mathcal{T}$, a process M is defined by $M_t(\omega) := \sum_{s \in \mathcal{T}_+} X_s(\omega)$.
3. The process Y is defined by

$$Y_t(\omega) := y_0 + \nu t + \sigma M_t(\omega)$$

where y_0, ν and $\sigma >= 0$ are constants.

Markov Property I

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a fixed function. Define a function $g : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ by

$$g(0, y) := f(y),$$
$$g(t+, y) := pg(t, y + \nu\delta + \sigma\sqrt{\delta}) + (1 - p)g(t, y + \nu\delta - \sigma\sqrt{\delta})$$

Theorem

Let m be an idempotent $\mathbb{F}$-market time. For any $s \geq t$,

$$\mathbb{E}^{\mathbb{P}}[f(Y_s) \mid \mathcal{F}_t^m] = g(s - m_t, Y_{m_t}).$$

Markov Property II

Corollary

Let m be an idempotent $\mathbb{F}$-market time. For any $s \geq t$,

$$\mathbb{E}^{\mathbb{P}}[f(Y_s) \mid \mathcal{F}_t^m] = \mathbb{E}^{\mathbb{P}}[f(Y_s) \mid m_t, Y_{m_t}].$$

Conclusions

1. Provide natural time delay examples that GJZ model is hard to accept.
2. Introduce a more general delayed process, called *market time* that does not require its random times are stopping times.
3. Introduce a class of market times, called *idempotent* market times, which contains many practically interesting market times. Give a characterization of idempotent market times.

Conclusions (cont.)

1. Introduce a delayed filtration called *market filtration* that is the filtration modulated by a given market time.
2. Show our market filtration is a natural extension of GJZ's continuously delayed filtration.
3. Investigate market filtrations in a binomial setting. Identify a Markov property for a conditional expectation given the market filtration.

The End

Thank you for your attention.